COMS 4995-3: Advanced Algorithms April 5, 2017

Lecture 19 – Strong Duality

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1 Introduction

Last time in class, we proved that a weak relation between Primal and Dual, namely that $v^* \geq w^*$. Today, we will show that Dual of the Dual is Primal and also prove the Strong Duality, i.e. $v^* = w^*$.

1.1 Last time

Duality:

$$\begin{aligned}
&Primal: \\
&\quad v^* = \min c^T x \\
&\quad\quad Ax = b \\
&\quad\quad x \geq 0 \\
&Dual: \\
&\quad w^* = \max b^T y \\
&\quad\quad A^T y \leq c \\
&\quad\quad u \in \mathbb{R}^m
\end{aligned}$$

Claim: $v^* \geq w^*$

1.2 Strong Duality

We would like to prove strong duality, i.e. $v^* = w^*$. The intuition is that the dual of dual will give us primal again.

Claim 1. *Dual(Dual) = Primal*

Proof.

$$\begin{aligned}
&Dual: \\
&\quad -\min(-b^T)y \\
&\quad A^T y \leq c
\end{aligned}$$

define $y = y^+ - y^-$, where $y^+, y^- \geq 0$. This allows us to say $A^T y + \delta = c$ where $\delta \geq 0$ as well.

Dual :

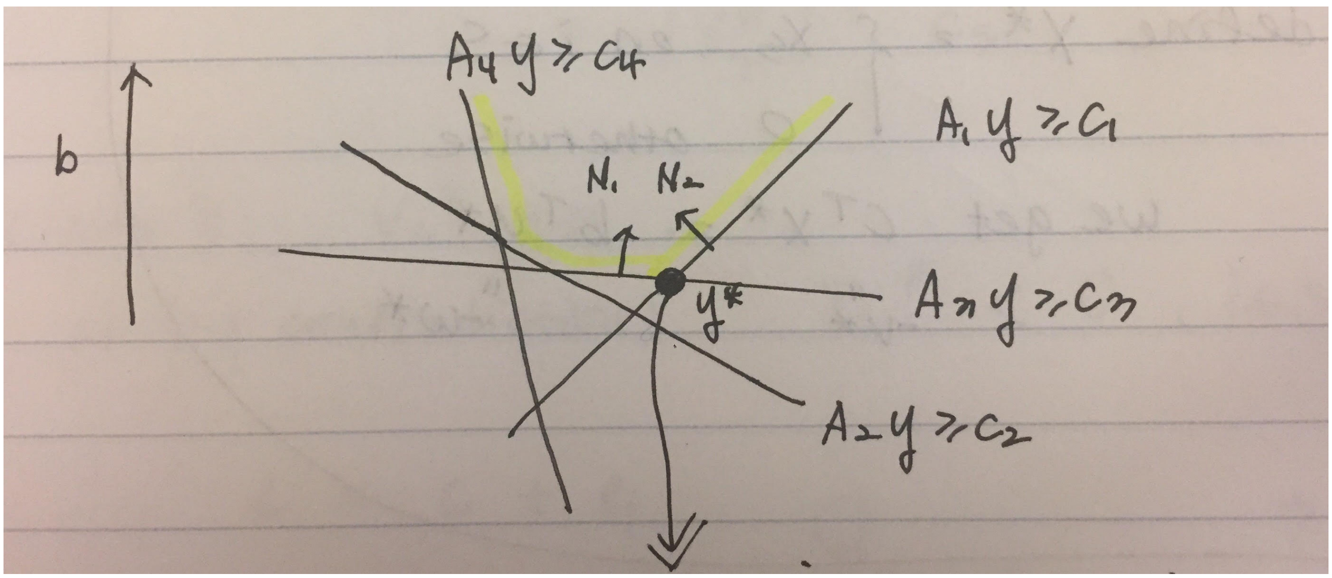
$$
\begin{aligned}
&-\min(-b^T)\cdot(y^+ - y^-) \\
&A^T(y^+ - y^-) + \delta = c \qquad\qquad \text{with } y^+, y^-, \delta \geq 0, \text{where } y^+, y^- \in \mathbb{R}^m, \delta \in \mathbb{R}^n
\end{aligned}
$$

Dual(*Dual*) :

$$
\begin{aligned}
&-\max(-c^T)x \\
\text{s.t.}\quad &-Ax \leq -b \\
&Ax \leq b \\
&x \leq 0 \qquad \text{which is the exact conditionals for Primal}
\end{aligned}
$$

$\square$

Intuition: Flip the signs of y and $c \Leftrightarrow$ Dual: $\min b^T y$, $A^T y \geq c$



Consider this as a ball being dropped into a convex bowl and it is pulled down by gravity. We know that the corner point of the convex bowl is always the optimal solution, and that the ball will be stuck at the point in which potential energy is minimized (and not move further down due to normal force pushing it back up). Here specifically in the picture, we note that $N_1 = A_3, N_2 = A_1$.

Thus, at equilibrium $\exists x_i$ (fraction of forces) s.t.

- b $= \Sigma_i A_i \cdot x_i = A_x$, where A_i corresponds to the columns of A
- $x_i \geq 0$ (all normal forces have to push upwards)

Here let's analyze the optimality. If $A_i y^* > c_i$, this means $x_i = 0$ (the particular constraint does not contribute as a corner to the solution). i.e. $(c_i - A_i y^*) \cdot x_i = 0$
$\therefore -v^* = c^T \cdot x = \Sigma_i c_i x_i = \Sigma_i x_i a_i y^* = -b^T \cdot y^* = -b^T \cdot y^* = -w^*$

1.3 Proof for Strong Duality

In this section, we will prove the strong duality i.e $v^* = w^*$.

Claim 2. $v^* = w^*$

Proof. We will flip the signs of y and c, so our objective function becomes

$$\min_{A^T y \geq c} b^T y$$

Let $y^* = \arg\min_{A^T y \leq c} b^T y$. Let us define a set S as the following,

$$S \triangleq \{\text{The set of constraints which are tight and linearly independent}\}$$

Note that $|S| < m$. Also, let A_s, c_s, x_s denote the elements of A, c and x restricted to set S. We have the following,

$$A_s^T y^* = c_s \text{By definition of set } S$$
$$b^T y^* = \min\{b^T y | A_s^T y^* \geq c_s\}$$

We will prove that $\exists\ x^*$ which satisfies the following

1. $A_s x_s^* = b$
2. $x^* \geq 0$
3. $c_s^T x_s^* = b^T y^*$

Claim 3. *$\exists\ x^*$ such that $A_s x_s^* = b$*

Proof. We will prove this by contradiction. Suppose that $\nexists\ x^*$ such that $A_s x_s^* = b$.
This implies $\exists\ z \in \mathbf{R}^m$ such that (1) $A_s^T z = 0$ (2) $b^T z < 0$.
Now, consider $y' = y^* + z$. y' is feasible in the dual because $A_s^T y' = A_s^T + A_s^T z = A_s^T y^*$.
Also, y' has better objective value

$$b^T y = b^T y^* + b^T z < b^T y^*$$

This is a contradiction because y^* is an optimal value. Hence, $\exists\ x^*$ such that $A_s x_s^* = b$ □

Claim 4. $b^T y^* = c^T x^*$

Proof.

$$\begin{aligned} b^T y^* &= (A_s x_s^*)^T y^* \\ &= (x_s^*)^T A_s^T y^* \\ &= (x_s^*)^T c_s \end{aligned}$$

Hence proved. $\square$

Claim 5. $x_s^* \geq 0$

Proof. We will prove this contradiction.

Assume that $x_i^* < 0 \ \forall i \in S$. Let c' be the new constraint defined as

$$c' = c + e_i \text{ where } e_i \text{ is vector with } \epsilon > 0 \text{ in } i^{th} \text{ position and zero elsewhere}$$

From above we get $c' = c_s + e_i$.

The equation $A_s^T y' = c_s'$ is solvable because it has $|S|$ linearly independent rows. Since $A_s y' = c_s' \geq c_s$, y' is feasible. The objective value at y' is

$$\begin{aligned}
b^T y' &= (A_s x_s^*)^T y' \\
&= (x_s^*)^T c_s' \\
&= (x_s^*)^T (c_s + e_i) \\
&= (x_s^*)^T c_s + (x_s^*)^T e_i \\
&< (x_s^*)^T c_s = b^T y^* \text{ (Because } x_s^* e_i \text{ is negative)}
\end{aligned}$$

This is a contradiction because y^* is an optimal value. Hence, $x_s^* \geq 0$ $\square$

This completes the proof for strong duality. $\square$